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Shock waves in a chain of two-level atoms with exchange and dipole-dipole interactions

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We use a small-amplitude multiple scale expansion to investigate the existence of shock waves in a chain of two-level atoms with both exchange and dipole-dipole interactions. We show that the exchange interaction allows the formation of the system of both bright and dark shock waves. Conversely, the dipole-dipole interaction results in the instability of the background and, as a consequence, in the prevention of the formation of shock waves. The analytical results are found to be in good qualitative agreement with a direct numerical integration of the system. [S1063-651X(97)06511-2]

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I. INTRODUCTION

Among possible excitations which arise in nonlinear lattices, localized modes, like solitons in integrable models, envelope solitons, and intrinsic localized modes in nonintegrable chains, have received a great deal of attention during the past years. This is justified by the relevance of these excitations in practical applications involving the transfer of energy along chains. On the other hand, in recent publications [1–7], it has been shown that nonlinear lattices may support other kinds of excitation which, at the initial stages of their evolution, display dynamics similar to that of shock waves in liquids and gases. Such shock waves have been observed in both integrable [1,2,4] and nonintegrable [2,5–7] lattices. The former case also allows an analytical description of the process after the shock wave has been formed. Thus, for example, in the Toda lattice it has been shown that the wave front of the developed shocks is a train of (bright) solitons. It is remarkable that this phenomenon survives in nonintegrable cases, i.e., also in these more general cases, shock waves develop in a series of localized pulses which can be viewed (in some approximation) as a train of solitons [7]. Moreover, it has been shown that discrete lattices may also support quite unusual *dark* shock waves [7] (dark shock waves in discrete systems were introduced in Ref. [8], and in a continuous dissipative system in Ref. [9]). This is the case, for example, of the so-called deformable [10] discrete nonlinear Schrödinger (DNLS) equation, for which shock waves of both types, bright and dark, have been found [7]. An analytical description of the initial stages of shock formation can be obtained from a small-amplitude multiscale expansion. As a matter of fact, the idea of finding a region in the parameter space where shock waves are expected is fairly simple: one has to look for the region in parameter space where the linear excitations of the background become effectively dispersionless. This analytical approach is quite general and, in principle, can be applicable to many discrete systems. One could expect, on the basis of these considerations, that shock wave formation should be a generic phenomenon to be observed in various nonlinear lattice systems.

The aim of the present paper is to investigate the existence of shock waves in a physical system consisting of a chain of two-level atoms. The system is described by a DNLS-like equation and includes both exchange and dipole-dipole interactions [11]. As a result, we find that the exchange interaction allows the formation of shock waves of both types, bright and dark. Conversely, the dipole-dipole interaction is quite destructive with respect to shock formation, since it results in the instability of the background at nonzero wave vectors. We give an analytical description of these phenomena in terms of a small-amplitude multiscale expansion. Our analytical results are found to be in good agreement with direct numerical integration of the system.

II. MODEL

We consider a one-dimensional chain of two-level atoms described by the Hamiltonian

$$H=\sum_n \mathcal{E}\hat{\sigma}_n^z-\tfrac{1}{2}\sum_{\langle m,n\rangle}[J_e(n,m)(\hat{\sigma}_n^+\hat{\sigma}_m^-+\hat{\sigma}_n^-\hat{\sigma}_m^+)+2J_d(n,m)\hat{\sigma}_n^x\hat{\sigma}_m^x-I(n,m)(\hat{\sigma}_n^z+\tfrac{1}{2})(\hat{\sigma}_m^z+\tfrac{1}{2})]. \quad (1)$$

Here $\hat{\sigma}_n=(\hat{\sigma}_n^x,\hat{\sigma}_n^y,\hat{\sigma}_n^z)$ are the Pauli spin operators; $\hat{\sigma}_n^{\pm}=\hat{\sigma}_n^x\pm i\hat{\sigma}_n^y$ are identified as creation ($\hat{\sigma}_n^+$) and annihilation ($\hat{\sigma}_n^-$) operators; $\mathcal{E}$ ($\mathcal{E}>0$) is the excitation energy of the two-level atom, $I(n,m)$, $J_e(n,m)$, and $J_d(n,m)$ describe the exciton-exciton, exchange, and dipole-dipole interactions between an exciton at the site n and that at the site m, respectively, and $\langle m,n\rangle$ means the sum over all m and n such that $m\neq n$.

We introduce the SU(2) coherent state representation for the Pauli spin operators,

$$|\mu_n\rangle=\frac{\exp(\mu_n\hat{\sigma}_n^+)}{\sqrt{1+|\mu_n|^2}}|-\rangle_n, \quad (2)$$

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where μ_n are the complex field variables, the symbols $|\pm\rangle_n$ stand for eigenstates of the angular momentum in which the eigenvalues of $\hat{\sigma}_n^z$ are $\pm\frac{1}{2}$, and the absolute value of $\hat{\sigma}_n$ is $\frac{1}{2}$. Then applying the stationary phase approximation for the path-integral formalism and applying the approximation of the nearest-neighbor interaction

$$J_{e,d}(n,m)=J_{e,d}(\delta_{n,m+1}+\delta_{n,m-1}), \tag{3}$$

$J_{e,d}$ being positive constants and similar formula valid for $I(n,m)$, we arrive at the equation of motion in the Lagrangian form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\mu}_n}-\frac{\partial L}{\partial\mu_n}=0, \tag{4}$$

where

$$L=\frac{i\hbar}{2}\sum_n\frac{1}{1+|\mu_n|^2}\left(\overline{\mu_n}\frac{d\mu_n}{dt}-\mu_n\frac{d\overline{\mu_n}}{dt}\right)-\langle\Lambda|H|\Lambda\rangle, \tag{5}$$

and $|\Lambda\rangle=\Pi_n|\mu_n\rangle$ is the coherent state of the whole spins.

Then, using the relations

$$\langle\mu_n|\hat{\sigma}_n^x|\mu_n\rangle=\frac{1}{2}\frac{\mu_n+\overline{\mu_n}}{1+|\mu_n|^2}, \tag{6a}$$

$$\langle\mu_n|\hat{\sigma}_n^y|\mu_n\rangle=\frac{i}{2}\frac{\overline{\mu_n}-\mu_n}{1+|\mu_n|^2}, \tag{6b}$$

$$\langle\mu_n|\hat{\sigma}_n^z|\mu_n\rangle=-\frac{1}{2}\frac{1-|\mu_n|^2}{1+|\mu_n|^2} \tag{6c}$$

we obtain by straightforward algebra the explicit form of the equation of motion

$$\begin{aligned}i\hbar\frac{d\mu_n}{dt}&=\mathcal{E}\mu_n-J\left(\frac{\mu_{n+1}-\mu_n^2\overline{\mu_{n+1}}}{1+|\mu_{n+1}|^2}+\frac{\mu_{n-1}-\mu_n^2\overline{\mu_{n-1}}}{1+|\mu_{n-1}|^2}\right)\\&\quad-J_d\left(\frac{\overline{\mu_{n+1}}-\mu_n^2\mu_{n+1}}{1+|\mu_{n+1}|^2}+\frac{\overline{\mu_{n-1}}-\mu_n^2\mu_{n-1}}{1+|\mu_{n-1}|^2}\right)\\&\quad+I\left(\frac{\mu_n|\mu_{n+1}|^2}{1+|\mu_{n+1}|^2}+\frac{\mu_n|\mu_{n-1}|^2}{1+|\mu_{n-1}|^2}\right).\end{aligned} \tag{7}$$

Here $J=J_e+J_d$.

III. BACKGROUND AND ITS STABILITY

The shock waves we are dealing with evolve against a carrier wave (cw) background of finite amplitude. Naturally the explicit form and the stability of the cw play a prominent role in the theory. It turns out that there are two essentially different cases: (i) J_d is either zero or small enough, and (ii) $J_d\neq0$. We shall see that the shock waves can exist only in the first case (see below).

Let us start with the case when $J_d=0$ (evidently $J=J_e$), i.e., with the case when only exchange and exciton-exciton interactions are presented. Then it is not difficult to ensure that

$$\mu_n^{(0)}=\rho e^{-i\omega t+ikn} \tag{8}$$

is a solution of Eq. (7) if

$$\hbar\omega=\mathcal{E}-2\cos(k)J\frac{1-\rho^2}{1+\rho^2}+2I\frac{\rho^2}{1+\rho^2}. \tag{9}$$

In order to study the stability of the cw, we make a substitution

$$\mu_n=(1+\psi_n)\rho e^{-i\omega t+ikn}, \tag{10}$$

where $|\psi_n|\ll|\mu_n|$ in Eq. (7), and linearize it. The dispersion relation $\Omega(K)$ associated with the thus obtained linear equation for ψ_n [$\propto\exp(-i\Omega t+iKn)$] reads

$$\begin{aligned}\hbar\Omega&=2\frac{1-\rho^2}{1+\rho^2}J\sin(k)\sin(K)\\&\quad\pm\frac{2\sqrt{2}}{1+\rho^2}J\sin\left(\frac{K}{2}\right)\left\{\cos^2(k)(1+\rho^2)^2\right.\\&\quad\left.-\left[\cos^2(k)(1-\rho^2)^2-2\frac{I}{J}\cos(k)\rho^2\right]\cos K\right\}^{1/2}.\end{aligned} \tag{11}$$

It is readily seen from here that the cw background is stable (i.e., Ω is real at all K), if

$$I<2J|\cos k|. \tag{12}$$

It is this region of the parameters to which we restrict our consideration in what follows.

In the present analysis k plays a part of a parameter. It is interesting to mention that the point $k=\pi/2$ corresponds to the less stable background which is destroyed by arbitrary exciton-exciton interaction. In what follows we restrict our analysis to the stable case.

As mentioned before, the region of the parameters where shock waves occur corresponds to a weak dispersion of the linear excitations against the background. In order to find this region we consider the behavior of Ω at small K. By the direct expansion we obtain

$$\begin{aligned}\hbar\Omega&=\frac{2J}{1+\rho^2}\left[(1-\rho^2)\sin k\pm\rho\sqrt{2\cos^2k+\widetilde{I}\cos k}\right]K\\&\quad-\frac{J}{3(1+\rho^2)}\left[(1-\rho^2)\sin k\pm\frac{1}{4}\sqrt{2\cos^2k+\widetilde{I}\cos k}\right.\\&\quad\left.\times\left(\rho-\frac{3}{\rho}\frac{(1-\rho^2)^2\cos k-2\rho^2\widetilde{I}}{2\cos k+\widetilde{I}}\right)\right]K^3+O(K^5).\end{aligned} \tag{13}$$

For the sake of convenience here we have introduced a notation $\widetilde{I}=I/J$. It follows from relation (13) that at

$$(1-\rho^2)\sin k = \mp \tfrac{1}{4}\sqrt{2\cos^2 k + \widetilde{I}\cos k}\times\left(\rho - \frac{3}{\rho}\,\frac{(1-\rho^2)^2\cos k - 2\rho^2\widetilde{I}}{2\cos k+\widetilde{I}}\right), \tag{14}$$

the group velocity dispersion becomes anomalously small $d^2\Omega/dK^2 = O(K^3)$ [while in all other regions of the parameters, $d^2\Omega/dK^2 = O(K)$].

In the case $J_d \neq 0$ the background can be written in the form

$$\mu_n^{(0)} = \nu\kappa^n \rho_{\nu\kappa}, \tag{15}$$

where the constant amplitude $\rho_{\nu\kappa}$ is given by

$$\rho_{\nu\kappa} = \left(\frac{2J_e + 2(1+\nu^2)J_d - \kappa\mathcal{E}}{2J_e + 2(1+\nu^2)J_d + \kappa\mathcal{E} + \kappa I}\right)^{1/2}; \tag{16}$$

ν is equal to 1 or i and $\kappa = \pm 1$. The parameter ν can be associated with the polarization, since $\sigma_n^x = 0$ at $\nu = i$ and $\sigma_n^y = 0$ at $\nu = 1$ ($\sigma_n^{x,y}$ being the eigenvalues of the respective operators), while κ describes a phase mismatch between two neighbors ($\kappa = 1$ and -1) corresponding to in-phase and out-of-phase cw, and can be associated with the center and the boundary of the Brillovin zone (BZ), respectively. In the case at hand the background is characterized by the relatively small energy of the excitation. That is, there must be

$$\mathcal{E} < 2J_e + 2(1+\nu^2)J_d - \tfrac{1}{2}(1-\kappa)I. \tag{17}$$

In order to examine the stability of the background [Eq. (16)] we linearize Eq. (3) about solution (15). The dispersion relation $\Omega(K)$ of the respective linear waves reads

$$\hbar^2\Omega(K)^2 = 4\left\{J(1-\cos K) + J_d\left[1 + \left(\frac{1-\rho_{\nu\kappa}^2}{1+\rho_{\nu\kappa}^2}\right)^2 \cos K\right]\right\} \times\left\{J_d(1-\cos K) + J\left[1 - \left(\frac{1-\rho_{\nu\kappa}^2}{1+\rho_{\nu\kappa}^2}\right)^2 \cos K\right] + 2I\kappa\cos K\,\frac{\rho_{\nu\kappa}^2}{(1+\rho_{\nu\kappa}^2)^2}\right\}. \tag{18}$$

The right-hand side of this expression is positively defined, and hence the background is stable, at $I < 2J$. This is in accordance with condition (12), which hereafter is understood in the generalized sense, i.e., is applicable to both cases (in the last one k must be taken either 0 or π). Moreover, from Eq. (13) we see that the dipole-dipole interaction introduces a gap into the spectrum in the center of the BZ. This is a destructive feature of shock wave formation, since it drastically increases the group-velocity dispersion. We expect, therefore, that shock waves should not exist for $J_d \neq 0$. Below, however, we will see that the even more destructive factor is the instability of the background at nonzero wave vectors.

IV. SHOCK WAVES IN THE SMALL-AMPLITUDE LIMIT

Let us start with the case $J_d = 0$. In order to describe shock waves we employ the small-amplitude expansion, in accordance with which

$$\mu_n = (\rho + a_n)\exp[i(-\omega t + kn - \phi_n)]. \tag{19}$$

Two real quantities a_n and ϕ_n are considered, depending on the slow variables $X = \gamma n$, $T = \gamma t$, and $\tau = \gamma^3 t$, γ being a small parameter, $\gamma \ll 1$, and are represented in a form of the sets

$$a_n = \gamma^2 a_n^{(0)} + \gamma^4 a_n^{(1)} + \cdots, \quad \phi_n = \gamma\phi_n^{(0)} + \gamma^3\phi_n^{(1)} + \cdots. \tag{20}$$

Collecting all terms of the same order, we arrive at a series of equations. In zero order we recover the dispersion relation (9). In the second and third orders of γ we arrive at the equations as follows

$$\hbar\frac{\partial\phi^{(0)}}{\partial T} = \frac{8\rho J\cos k}{(1+\rho^2)^2}a^{(0)} + \frac{4\rho}{(1+\rho^2)^2}Ia^{(0)} - 2\sin(k)J\frac{1-\rho^2}{1+\rho^2}\frac{\partial\phi^{(0)}}{\partial X}, \tag{21}$$

$$\hbar\frac{\partial a^{(0)}}{\partial T} = \rho J\cos(k)\frac{\partial^2\phi^{(0)}}{\partial X^2} - 2\sin(k)J\frac{1-\rho^2}{1+\rho^2}\frac{\partial a^{(0)}}{\partial X}. \tag{22}$$

Let us introduce new variables $(\xi_\pm, T)$ instead of (X,T), where $\xi_\pm = X - c_\pm T$ and $c_\pm$ makes a sense of velocity:

$$c_\pm = \frac{2J}{\hbar(1+\rho^2)}\left[(1-\rho^2)\sin k \pm \rho\sqrt{2\cos^2 k + \widetilde{I}\cos k}\right]. \tag{23}$$

Comparing this result with Eq. (13), one ensures that $c_\pm = d\Omega_\pm/dK$ at $K=0$, i.e., $c_\pm$ are group velocities of two branches of the spectrum in the center of the BZ. Then it follows from Eqs. (21) and (22) that $a^{(0)} = a^{(0)}(\xi_\pm) = a_\pm$ and $\phi^{(0)} = \phi^{(0)}(\xi_\pm)$ are solutions and the relation between them is given by

$$a_\pm = \mp(1+\rho^2)\frac{\sqrt{\cos k}}{2\sqrt{2\cos k + \widetilde{I}}}\frac{\partial\phi^{(0)}}{\partial\xi_\pm}. \tag{24}$$

The equations appearing in the forth and fifth orders of γ are given in the Appendix. The condition of their compatibility, Eq. (A6), can be written down in the form of the Korteweg–de Vries (KdV) equation:

$$\frac{\partial a_\pm}{\partial\tau} + \alpha(k)a_\pm\frac{\partial a_\pm}{\partial\xi_\pm} + \beta(k)\frac{\partial^3 a_\pm}{\partial\xi_\pm^3} = 0. \tag{25}$$

Here

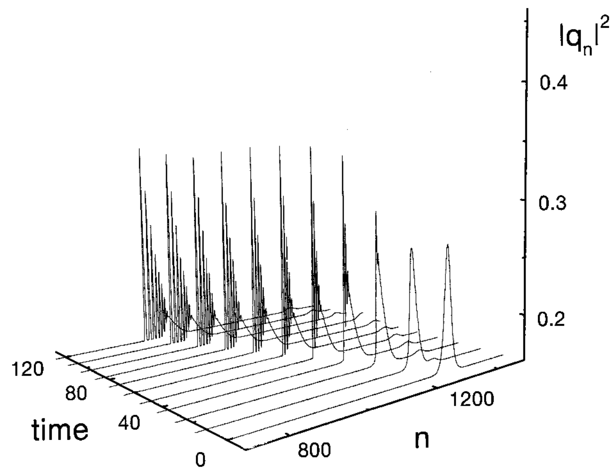

FIG. 1. Evolution of a bright shock against a nonzero background with $k=0$ and $\rho=0.4$, and for parameter values $I=3.0$, $E=1.0$, and $J_d=0$, with J_e determined from Eq. (14).

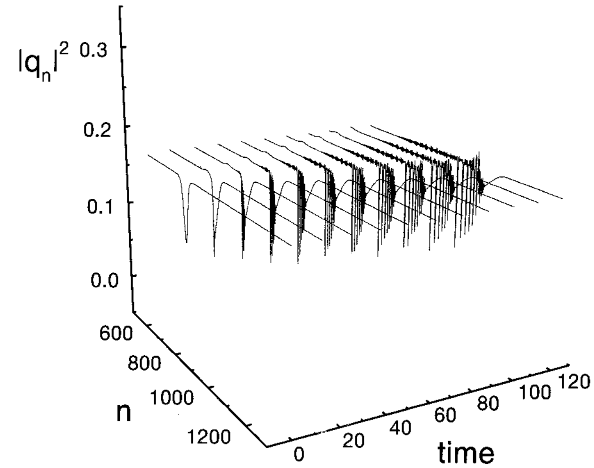

FIG. 2. Evolution of a dark shock against nonzero background with $k=0$ and $\rho=0.4$, and for parameter values $I=3.0$, $E=1.0$, and $J_d=0$, with J_e determined from Eq. (14).

$$\alpha(k)=\frac{4J}{\hbar(1+\rho^2)^2}\Bigg[-10\rho\sin k\mp(3\rho^2-1)\times\sqrt{2\cos^2 k+\tilde{I}\cos k}\pm\frac{(3-\rho^2)\sqrt{\cos k}(2\cos k+\tilde{I})}{2\sqrt{2\cos k+\tilde{I}}}-2\rho\tilde{I}\tan k\Bigg], \tag{26}$$

$$\beta(k)=\frac{2}{3}\frac{J}{\hbar(1+\rho^2)}\Bigg\{(1-\rho^2)\sin k\pm\tfrac{1}{4}\sqrt{2\cos^2 k+\tilde{I}\cos k}\times\Bigg[\rho-\frac{3}{\rho}\frac{(1-\rho^2)^2\cos k-2\tilde{I}\rho^2}{2\cos k+\tilde{I}}\Bigg]\Bigg\}. \tag{27}$$

It follows from Eqs. (25) and (27) that, if Eq. (14) is satisfied, the coefficient $\beta(k)$ becomes zero and the KdV equation is reduced to the well-known equation

$$\frac{\partial a_\pm}{\partial\tau}+\alpha(k)a_\pm\frac{\partial a_\pm}{\partial\xi_\pm}=0, \tag{28}$$

which in our case describes initial stages of the evolution of a shock wave in a chain of two-level atoms with the energy transfer by the exchange interaction. To check this result, we numerically integrated Eq. (7) on a long chain (long enough to neglect the influence of boundary conditions) by taking as the initial condition a bell-shaped bright or dark pulse of the type

$$\mu_n=\rho e^{ikn}\left(1\pm\frac{A}{\cosh[(n-n_0)]^2}\right). \tag{29}$$

In Figs. 1 and 2 we show the time evolution of initial bright and dark pulses, respectively, of amplitude $|A|=0.12$, on an in-phase background $(k=0)$ with $\rho=0.4$, for parameter values $I=3.0$, $E=1.0$, and $J_d=0$, with J_e determined from Eq. (14). We see that the initial profiles bend forward in the direction of propagation until reaching a breaking time at which oscillations, starting from the top, develop on the profiles. These oscillations can be viewed as a train of, respectively, bright and dark pulses ordered with decreasing amplitude when moving from the front to the rear of the wave. Depending on parameter values, however, the train decomposition can occur after times so long that the shock may be considered effectively stable for all practical purpouses. This is shown in Fig. 3 where a bright shock profile is reported after an evolution time of 1000. From this figure we see that after such a time the shock front is still quite evident, and the oscillations are closely packed behind it. It is remarkable that, in spite of the different types of nonlinearity characterizing our system, the waves which develop are very similar to the bright and dark shocks observed in the deformable DNLS system [7]. This shows that the exchange interaction can support the formation of shocks. Conversely, the previous analysis predicted that the strong dispersion induced by the dipole-dipole interaction should prevent the formation of shock waves. This phenomenon can be easily checked by

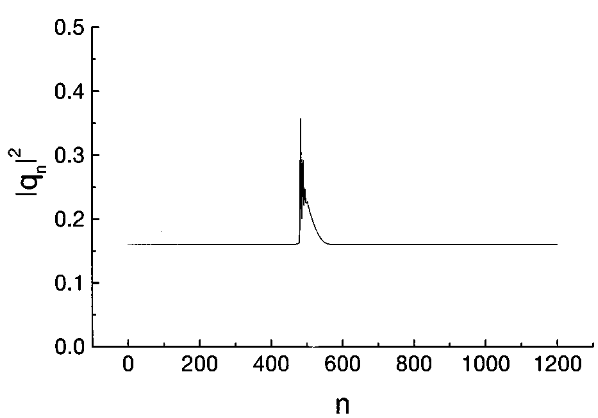

FIG. 3. A bright shock profile after an evolution time of 1000 for parameter values $k=0$, $\rho=0.4$, $I=0.1$, $E=1.0$, and $J_d=0$, and with J_e determined from Eq. (14).

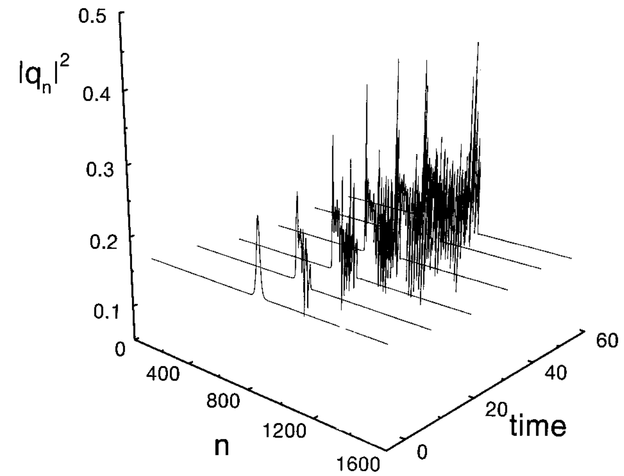

FIG. 4. The same time evolution as in Fig. 1 in the case of nonzero dipole-dipole interaction: $J_d = 0.025$.

direct numerical simulations as shown in Figs. 4 and 5. In these figures we have reported the time evolution of the same initial pulses and at the same parameter values as in Figs. 1 and 2, but now with $J_d = 0.025$. From these figures it is clear that the shock waves are destroyed even for a small value of J_d.

To better understand this phenomenon, we shall consider the case of small J_d values by assuming that $J_d = \gamma^3 \widetilde{J}_d$. Within the framework of this scaling, we can provide a small-amplitude multiscale expansion similar to the one described above but for the case of a background with $k=0$ and with zero frequency [i.e., for the background given by Eqs. (15) and (16)]. We drop details of calculations [see Eqs. (A7) and (A8) in the Appendix] and present just the final form of the evolution equation,

$$\frac{\partial a_\pm}{\partial \tau} + \alpha_\kappa a_\pm \frac{\partial a_\pm}{\partial \xi_\pm} + \nu^2 \zeta_\kappa a_\pm + \nu^2 \delta_\kappa \phi^{(0)2} = 0. \tag{30}$$

Here $\phi^{(0)}$ is linked with $a_\pm$ by relation (24),

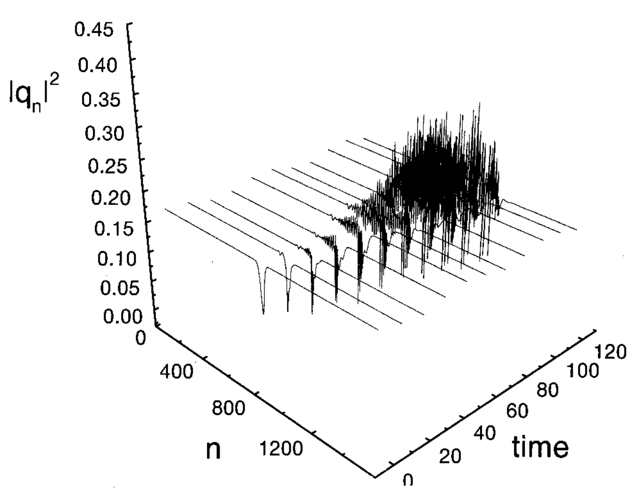

FIG. 5. The same time evolution as in Fig. 2 in the case of nonzero dipole-dipole interaction: $J_d = 0.025$.

$$\zeta_\kappa = -4 \frac{\widetilde{J}_d}{\hbar} \frac{1 + 6\rho^2 + \rho^4}{\rho(1+\rho^2)^3}, \tag{31}$$

$$\delta_\kappa = -8 \frac{\widetilde{J}_d}{\hbar} \frac{1-\rho^2}{(1+\rho^2)^2}, \tag{32}$$

$\alpha_1 = \alpha(0)$ and $\alpha_{-1} = \alpha(\pi)$.

In the special case $\rho = 1$, Eq. (30) takes the form

$$\frac{\partial a_\pm}{\partial \tau} + \alpha_\kappa a_\pm \frac{\partial a_\pm}{\partial \xi_\pm} + \nu^2 \zeta_\kappa a_\pm = 0\,. \tag{33}$$

It is seen from this equation that the dipole-dipole interaction results in an effective dissipative or amplfying term which leads either to a decrease or increase of the amplitude of the shock wave. This is a reflection of the fact that the dipole-dipole interaction leads to an instability of the background at any wave vector nonequal to zero. Thus the mechanism of the destruction of the shock wave in the case at hand can be described in the follwing way. The nonlinearity results in a self-phase modulation, and in particular the phase mismatch between the nearest neighbors changes with the amplitude of the wave [see Eq. (24)]. Hence the change of the wave amplitude results in the change of the spectrum: the contribution of the harmonics with $k \neq 0$ rises. That is, these harmonics make the wave unstable at $J_d \neq 0$. Recall that the stability of a background only with $k = 0, \pm\pi/2, \pi$ was proven in Sec. III, while backgrounds with other wave vectors are unstable.

V. CONCLUSION

It has been shown that the exchange interaction in a chain of two-level atoms described by a DNLS-like equation allows the formation of shock waves of both types, bright and dark. Conversely, dipole-dipole interaction is quite destructive with respect to shock formation, since it results in the instability of the carrier wave background for large domains of wave vectors. We provided an analytical desription of these phenomena in terms of a small-amplitude multiscale expansion, and compared the result with a direct numerical integration of the system, finding a good qualitative agreement.

Like some other localized excitations (for instance, solitons or intrinsic localized modes) shock waves are sufficiently long-living objects. However, after some time they decay in a train of solitonlike excitations. Then a natural question arises: why the newborn localized excitations do not produce ''secondary'' shock waves. The answer is in the fact that, during the evolution of the shock waves, the spectrum of the excitations is changed, i.e., characteristic wave numbers are shifted. This breaks down condition (14), i.e., the relation among parameters necessary for shock wave creation.

The results obtained here for the existence of bright and dark shock waves stems from the coherence properties of excitons in a system of two-level atoms which are described by Eq. (7), having the form of a classical equation. The situation here is somewhat similar to that in laser physics [12], where two-level atoms (but with atom-atom interaction omitted) in interaction with an intense radiation field are often described by classical equations. Exploiting the similarity between these two cases leads to the presumption that shock waves may be observed in exciton systems by applying an intense radiation field to the insulating solids or molecular crystals to which the Frencel exciton model can be applied. Then initial exciton profiles may be realizable by superimposing a strong pulse field with respect to space as well as time variables. Generally speaking, more attention, both theoretical and experimental, has so far been paid to Wannier excitons in exciton problems in solid-state physics. It would therefore be worthwhile to seek the existence of shock waves for the Wannier excitons, as well.

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APPENDIX

In the fourth and fifth orders of γ, one obtains the equations

$$\hbar\frac{\partial\phi^{(1)}}{\partial T}+2\sin(k)J\frac{1-\rho^2}{1+\rho^2}\frac{\partial\phi^{(1)}}{\partial X}=Aa^{(1)}+B, \qquad \text{(A1)}$$

$$\hbar\frac{\partial a^{(1)}}{\partial T}=\rho J\cos(k)\frac{\partial^2\phi^{(1)}}{\partial X^2}-2\sin(k)J\frac{1-\rho^2}{1+\rho^2}\frac{\partial a^{(1)}}{\partial X}+C, \qquad \text{(A2)}$$

where

$$A=\frac{4\rho(2J\cos k+I)}{(1+\rho^2)^2}, \qquad \text{(A3)}$$

$$B=\frac{\hbar c_{\pm}}{\rho}a^{(0)}\frac{\partial\phi^{(0)}}{\partial\xi_{\pm}}-\hbar\frac{\partial\phi^{(0)}}{\partial\tau}+J\frac{1-\rho^2}{1+\rho^2}\left\{\cos k\left[\frac{4(3-\rho^2)}{(1-\rho^4)(1+\rho^2)}a^{(0)2}-\frac{1}{\rho}\frac{1-\rho^2}{1+\rho^2}\frac{\partial^2a^{(0)}}{\partial\xi_{\pm}^2}+\left(\frac{\partial\phi^{(0)}}{\partial\xi_{\pm}}\right)^2\right]-\sin k\left(\frac{1}{6}\frac{\partial^3\phi^{(0)}}{\partial\xi_{\pm}^3}+\frac{1-4\rho^2-\rho^4}{\rho(1-\rho^4)}a^{(0)}\frac{\partial\phi^{(0)}}{\partial\xi_{\pm}}\right)\right\}+\frac{2I}{(1+\rho^2)^2}\left(\frac{3-\rho^2}{1+\rho^2}a^{(0)2}+\rho\frac{\partial^2a^{(0)}}{\partial\xi_{\pm}^2}\right), \qquad \text{(A4)}$$

$$C=2J\left[\cos k\left(\frac{1-\rho^2}{1+\rho^2}\frac{\partial a^{(0)}}{\partial\xi_{\pm}}\frac{\partial\phi^{(0)}}{\partial\xi_{\pm}}+\frac{\rho}{24}\frac{\partial^4\phi^{(0)}}{\partial\xi_{\pm}^4}\right)+\sin k\left(\rho\frac{\partial\phi^{(0)}}{\partial\xi_{\pm}}\frac{\partial^2\phi^{(0)}}{\partial\xi_{\pm}^2}-\frac{1}{6}\frac{1-\rho^2}{1+\rho^2}\frac{\partial^3a^{(0)}}{\partial\xi_{\pm}^3}+\frac{4\rho}{(1+\rho^2)^2}a^{(0)}\frac{\partial a^{(0)}}{\partial\xi_{\pm}}\right)\right]. \qquad \text{(A5)}$$

Here we have taken into account that $a^{(0)}$ and $\phi^{(0)}$ are functions only on $\xi_{\pm}$ and a ''slower'' time τ.

The condition of the compatibility of Eqs. (A1) and (A2) reads

$$\hbar\frac{\partial B}{\partial T}+2\sin(k)J\frac{1-\rho^2}{1+\rho^2}\frac{\partial B}{\partial X}+AC=0\,. \qquad \text{(A6)}$$

In order to take into account the effect of the dipole-dipole interaction $J_d=\gamma^3\tilde{J}_d$, one has to make a change in Eq. (A6)

$$B\mapsto B+4\,\overline{\nu\tilde{J}_d}\phi^{(0)}, \qquad \text{(A7)}$$

$$C\mapsto C+4\,\overline{\nu\kappa}\tilde{J}_d\rho\frac{1-\rho^2}{1+\rho^2}\phi^{(0)2}-2\,\overline{\nu\tilde{J}_d}\rho\frac{1-4\rho^2-\rho^4}{(1+\rho^2)^2}a^{(0)}. \qquad \text{(A8)}$$

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